MHD-Drift Equations: from Langmuir circulations to MHD-dynamo?

By V. A. Vladimirov

Dept of Mathematics, University of York, Heslington, York, YO10 5DD, UK

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We have derived the closed system of averaged MHD-equations for general oscillating flows, which are purely oscillating in the main approximation. We have used the mathematical approach which combines the two-timing method and the notion of the distinguished limit. Properties of the commutators are used to simplify calculations. The direct connection with a vortex dynamo (or the Langmuir circulations) has been demonstrated and a conjecture on the MHD-dynamo has been formulated.

1. Introduction

This paper derives the averaged MHD-equations for oscillating flows. The resulting equations are similar to the original MHD-equations, but surprisingly (instead of commonly expected Reynolds stresses) the *drift velocity* (or just *the drift*) plays a part of an additional advection velocity.

It is known that the drift can appear from either Lagrangian or Eulerian considerations. The *Lagrangian drift* appears as the average motion of Lagrangian particles and its theory is often based on the averaging of ODEs, see Stokes (1847), Lamb (1932), Longuet-Higgins (1953), Batchelor (1967), Andrews & McIntyre (1978), Craik (1982), Yudovich (2006). In this paper we focus on the *Eulerian drift*, which appears as the results of the Eulerian averaging of related PDEs without addressing the motion of particles, see Craik & Leibovich(1976), Craik (1985), Riley (2001), Vladimirov (2010), Ilin & Morgulis (2011). The detailed materials about the Eulerian drift can be found in Vladimirov (2010).

To derive the averaged equations we employ the two-timing method, see *e.g.* Nayfeh (1973), Kevorkian & Cole (1996). We expose it as an elementary, systematic, and justifiable procedure that follows the form developed by Yudovich (2006), Vladimirov (2005), Vladimirov (2008), Vladimirov (2010). This mathematical procedure is complemented by a novel material on the distinguished limit, which allows to find the proper slow time-scale.

2. Functions and operations

We introduce functions of variables $\boldsymbol{x}=(x_1,x_2,x_3)$, s, and τ, which in the text below serve as dimensionless cartesian coordinates, slow time, and fast time.

Definition 1. The class $\mathbb{H}$ of *hat-functions* is defined as

$$\widehat{f}\in\mathbb{H}:\quad \widehat{f}(\boldsymbol{x},s,\tau)=\widehat{f}(\boldsymbol{x},s,\tau+2\pi) \tag{2.1}$$

where the τ-dependence is always 2π-periodic; the dependencies on $\boldsymbol{x}$ and t are not specified.

Definition 2. For an arbitrary $\widehat{f} \in \mathbb{H}$ the *averaging operation* is

$$\langle \widehat{f} \rangle \equiv \frac{1}{2\pi} \int_{\tau_0}^{\tau_0+2\pi} \widehat{f}(\boldsymbol{x}, s, \tau)\, d\tau, \qquad \forall\ \tau_0 \tag{2.2}$$

where during the τ-integration $s = \mathrm{const}$ and $\langle \widehat{f} \rangle$ does not depend on τ_0.

Definition 3. The class $\mathbb{T}$ of *tilde-functions* is such that

$$\widetilde{f} \in \mathbb{T}: \quad \widetilde{f}(\boldsymbol{x}, s, \tau) = \widetilde{f}(\boldsymbol{x}, s, \tau + 2\pi), \quad \text{with} \quad \langle \widetilde{f} \rangle = 0, \tag{2.3}$$

The tilde-functions are also called purely oscillating functions ($\mathbb{T}$-*function* represents a special case of $\mathbb{H}$-function with zero average).

Definition 4. The class $\mathbb{B}$ of *bar-functions* is defined as

$$\overline{f} \in \mathbb{B}: \quad \overline{f}_\tau \equiv 0, \quad \overline{f}(\boldsymbol{x}, s) = \langle \overline{f}(\boldsymbol{x}, s) \rangle \tag{2.4}$$

(any $\mathbb{H}$-function can be uniquely separated into its $\mathbb{B}$- and $\mathbb{T}$- parts with the use of (2.2))

Definition 5. $\mathbb{T}$*-integration (or tilde-integration)*: for a given $\widetilde{f}$ we introduce a new function $\widetilde{f}^\tau$ called the $\mathbb{T}$-integral of $\widetilde{f}$:

$$\widetilde{f}^\tau \equiv \int_0^\tau \widetilde{f}(\boldsymbol{x}, s, \sigma)\, d\sigma - \frac{1}{2\pi} \int_0^{2\pi} \Big(\int_0^\mu \widetilde{f}(\boldsymbol{x}, s, \sigma)\, d\sigma \Big)\, d\mu \tag{2.5}$$

which represents the unique solution of a PDE $\partial \widetilde{f}^\tau / \partial \tau = \widetilde{f}$ (with an unknown function $\widetilde{f}^\tau$ and a known function $\widetilde{f}$) supplemented by the condition $\langle \widetilde{f} \rangle = \langle \widetilde{f}^\tau \rangle = 0$ (2.3).

The τ-derivative of $\mathbb{T}$-function always represents $\mathbb{T}$-function. However the τ-integration of $\mathbb{T}$-function can produce an $\mathbb{H}$-function. An example: $\widetilde{f} = \overline{f}_1 \sin \tau$ where $\overline{f}_1$ is an arbitrary $\mathbb{B}$: one can see that $\langle \widetilde{f} \rangle \equiv 0$, however $\langle \int_0^\tau \widetilde{f}(\boldsymbol{x}, s, \sigma) d\sigma \rangle = \overline{f}_1 \neq 0$, unless $\overline{f}_1 \equiv 0$. Formula (2.5) keeps the result of integration inside the $\mathbb{T}$-class. $\mathbb{T}$-integration is inverse to τ-differentiation $(\widetilde{f}^\tau)_\tau = (\widetilde{f}_\tau)^\tau = \widetilde{f}$; the proof is omitted.

Definition 6. A dimensionless function $f = f(\boldsymbol{x}, s, \tau)$ belongs to the class $\mathbb{O}(1)$

$$f \in \mathbb{O}(1) \tag{2.6}$$

if $f = O(1)$ and all required partial $\boldsymbol{x}$-, s-, and τ-derivatives of f are also $O(1)$.

Here we emphasize that through all the text below all large or small parameters are represented by various degrees of σ only; these parameters appear as explicit multipliers in all formulae containing tilde- and bar-functions; these functions always belong to $\mathbb{O}(1)$-class.

We also will use some properties of τ-derivatives such as

$$\widehat{f}_\tau = \widetilde{f}_\tau, \quad \langle \widehat{f}_\tau \rangle = \langle \widetilde{f}_\tau \rangle = 0 \tag{2.7}$$

The product of two $\mathbb{T}$-functions $\widetilde{f}$ and $\widetilde{g}$ represents a $\mathbb{H}$-function: $\widetilde{f}\widetilde{g} \equiv \widehat{F}$, say. Separating $\mathbb{T}$-part $\widetilde{F}$ from $\widehat{F}$ we write

$$\widetilde{F} = \widehat{F} - \langle \widehat{F} \rangle = \widetilde{f}\widetilde{g} - \langle \widetilde{f}\widetilde{g} \rangle = \{\widetilde{f}\widetilde{g}\} \tag{2.8}$$

where the notation $\{\cdot\}$ for the tilde-part is introduced to avoid two levels of tildes. We will use that the unique solution of a PDE inside the tilde-class is

$$\partial \widetilde{f} / \partial \tau = 0 \quad \Rightarrow \quad \widetilde{f} \equiv 0 \tag{2.9}$$

which follows from (2.5). Since the average operation (2.2) is proportional to the integration over τ, then from the integration by parts we have

$$\langle [\widetilde{\boldsymbol{a}}, \widetilde{\boldsymbol{b}}_\tau] \rangle = -\langle [\widetilde{\boldsymbol{a}}_\tau, \widetilde{\boldsymbol{b}}] \rangle = -\langle [\widetilde{\boldsymbol{a}}_\tau, \widehat{\boldsymbol{b}}] \rangle, \ \langle [\widetilde{\boldsymbol{a}}, \widetilde{\boldsymbol{b}}^\tau] \rangle = -\langle [\widetilde{\boldsymbol{a}}^\tau, \widetilde{\boldsymbol{b}}] \rangle = -\langle [\widetilde{\boldsymbol{a}}^\tau, \widehat{\boldsymbol{b}}] \rangle \tag{2.10}$$

where $[\boldsymbol{a},\boldsymbol{b}]$ stands for the commutator of two vector fields $\boldsymbol{a}$ and $\boldsymbol{b}$ which is antisymmetric and satisfies Jacobi's identity for vector fields $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$:

$$[\boldsymbol{a},\boldsymbol{b}] \equiv (\boldsymbol{b}\cdot\nabla)\boldsymbol{a} - (\boldsymbol{a}\cdot\nabla)\boldsymbol{b}, \tag{2.11}$$

$$[\boldsymbol{a},\boldsymbol{b}] = -[\boldsymbol{b},\boldsymbol{a}], \quad [\boldsymbol{a},[\boldsymbol{b},\boldsymbol{c}]] + [\boldsymbol{c},[\boldsymbol{a},\boldsymbol{b}]] + [\boldsymbol{b},[\boldsymbol{c},\boldsymbol{a}]] = 0 \tag{2.12}$$

A useful property of the commutator is

$$\operatorname{div}\boldsymbol{a} = 0, \quad \operatorname{div}\boldsymbol{b} = 0 \quad \Rightarrow \quad \operatorname{div}[\boldsymbol{a},\boldsymbol{b}] = 0 \tag{2.13}$$

For any tilde-function $\widetilde{\boldsymbol{a}}$ and bar-function $\overline{\boldsymbol{b}}$ (2.10),(2.12) give

$$\langle[\widetilde{\boldsymbol{a}},[\overline{\boldsymbol{b}},\widetilde{\boldsymbol{a}}^\tau]]\rangle = [\overline{\boldsymbol{b}},\overline{\boldsymbol{V}}_a] \quad \text{where} \quad \overline{\boldsymbol{V}}_a \equiv \frac{1}{2}\langle[\widetilde{\boldsymbol{a}},\widetilde{\boldsymbol{a}}^\tau]\rangle \tag{2.14}$$

3. Two-timing problem and distinguished limits

The governing equation for MHD-dynamics of a homogeneous inviscid incompressible fluid with velocity field $\boldsymbol{u}^*$, magnetic fields $\boldsymbol{h}^*$, vorticity $\boldsymbol{\omega}^* \equiv \nabla^*\times\boldsymbol{u}^*$ and current $\boldsymbol{j}^* \equiv \nabla^*\times\boldsymbol{h}^*$ is taken in the vorticity form

$$\begin{aligned}
&\frac{\partial\boldsymbol{\omega}^*}{\partial t^*} + [\boldsymbol{\omega}^*,\boldsymbol{u}^*]^* - [\boldsymbol{j}^*,\boldsymbol{h}^*]^* = 0, \quad \text{in} \quad \mathcal{D}^*\\
&\frac{\partial\boldsymbol{h}^*}{\partial t^*} + [\boldsymbol{h}^*,\boldsymbol{u}^*]^* = 0,\\
&\nabla^*\cdot\boldsymbol{u}^* = 0, \quad \nabla^*\cdot\boldsymbol{h}^* = 0
\end{aligned} \tag{3.1}$$

where asterisks mark dimensional variables, t^*-time, $\boldsymbol{x}^* = (x_1^*, x_2^*, x_3^*)$-cartesian coordinates, $\nabla^* = (\partial/\partial x_1^*, \partial/\partial x_2^*, \partial/\partial x_3^*)$, and $[\,\cdot\,,\cdot\,]^*$ stands for the dimensional commutator (2.11). In this paper we deal with the transformations of equations, so the form of flow domain $\mathcal{D}^*$ and particular boundary conditions can be specified at the later stages.

We accept that the considered class of (unknown) oscillatory solutions $\boldsymbol{u}^*,\boldsymbol{h}^*$ possesses characteristic scales of velocity U, magnetic field H, length L, and high frequency σ^*

$$U, \quad H, \quad L, \quad \sigma^* \gg 1/T; \quad T \equiv L/U \tag{3.2}$$

where T is a dependent time-scale. In the chosen system of units the dimensions of U and H coincide; we choose them being the same order $U = H$. The dimensionless variables and frequency are

$$\boldsymbol{x} \equiv \boldsymbol{x}^*/L, \quad t \equiv t/T, \quad \widehat{\boldsymbol{u}} \equiv \widehat{\boldsymbol{u}}^*/U, \quad \widehat{\boldsymbol{h}} \equiv \widehat{\boldsymbol{h}}^*/U, \quad \sigma \equiv \sigma^* T \gg 1 \tag{3.3}$$

We assume that the flow has its own intrinsic slow-time scale T_{slow} (which can be different from T) and consider solutions of (3.1) in the form of hat-functions (2.1)

$$\boldsymbol{u}^* = U\widehat{\boldsymbol{u}}(\boldsymbol{x},s,\tau), \quad \boldsymbol{h}^* = U\widehat{\boldsymbol{h}}(\boldsymbol{x},s,\tau); \quad \text{with } \tau \equiv \sigma t,\ s \equiv \Omega\, t,\ \Omega \equiv T/T_{\text{slow}} \tag{3.4}$$

Then the use of the chain rule and transformation to dimensionless variables give

$$\begin{aligned}
&\left(\frac{\partial}{\partial\tau} + \frac{\Omega}{\sigma}\frac{\partial}{\partial s}\right)\widehat{\boldsymbol{\omega}} + \frac{1}{\sigma}[\widehat{\boldsymbol{\omega}},\widehat{\boldsymbol{u}}] - \frac{1}{\sigma}[\widehat{\boldsymbol{j}},\widehat{\boldsymbol{h}}] = 0\\
&\left(\frac{\partial}{\partial\tau} + \frac{\Omega}{\sigma}\frac{\partial}{\partial s}\right)\widehat{\boldsymbol{h}} + \frac{1}{\sigma}[\widehat{\boldsymbol{h}},\widehat{\boldsymbol{u}}] = 0\\
&\operatorname{div}\widehat{\boldsymbol{h}} = 0, \quad \operatorname{div}\widehat{\boldsymbol{u}} = 0
\end{aligned} \tag{3.5}$$

In order to keep variable s 'slow' in comparison with τ we have to accept that $\Omega/\sigma \ll 1$.

Then eqn.(3.5) contains two independent small parameters:

$$\varepsilon \equiv \frac{1}{T\sigma^*} = \frac{1}{\sigma}, \quad \varepsilon_1 \equiv \frac{1}{T_{\text{slow}}\sigma^*} \equiv \frac{\Omega}{\sigma} \tag{3.6}$$

Here we must make an auxiliary (technically essential) assumption: after the use of the chain rule (3.5) variables s and τ are (temporarily) considered to be *mutually independent*:

$$\tau, \qquad s \qquad - \text{independent variables} \tag{3.7}$$

From the mathematical viewpoint the increasing of the number of independent variables in a PDE represents a very radical step, which leads to an entirely new PDE. This step should be justified *a posteriori* by the estimations of the error of the obtained solution (rewritten back to the original variable t) substituted to the original equation (3.1).

In a rigorous asymptotic procedure with $\sigma \to \infty$ one has to consider asymptotic paths on the $(\varepsilon, \varepsilon_1)$-plane such that

$$(\varepsilon, \varepsilon_1) \to (0, 0) \tag{3.8}$$

Each such path can be prescribed by a particular function $\Omega(\sigma)$. One may expect that there are infinitely many different (although some of them can coincide) solutions to (3.5) corresponding to different $\Omega(\sigma)$. However for these equations (as well as for many others) a unique path can be found, which is called the *distinguish limit* (or the *distinguished path*). The notion of the *distinguish limit* is practical and heuristic, see Nayfeh (1973), Kevorkian & Cole (1996); its definition can vary for different equations and in different books and papers. For our problem we write that *the distinguished limit is given by such a function $\Omega = \Omega_d(\sigma)$ that allows to build a self-consistent asymptotic solution.* Here the term *self-consistent asymptotic solution* means that the required successive approximations can be calculated. These calculations include the elimination of the reducible secular in s terms; the *reducible secular terms* are such terms which can be excluded by increasing the slow time-scale. For instance, a non-secular term proportional to $\sin s$ gives secular terms in its Taylor's decomposition with respect to $t = \sigma s$). Below we show that the choice

$$\Omega(\sigma) = 1/\sigma: \qquad \tau = \sigma t, \quad s = t/\sigma, \qquad \alpha = \text{const} \tag{3.9}$$

allows to build the distinguished limit solution. The uniqueness of such a path for the considered class of solutions can be proven but we avoid such details in this paper. Hence the governing equations are

$$\begin{aligned}
&\widehat{\boldsymbol{\omega}}_\tau + \varepsilon[\widehat{\boldsymbol{\omega}}, \widehat{\boldsymbol{u}}] - \varepsilon[\widehat{\boldsymbol{j}}, \widehat{\boldsymbol{h}}] + \varepsilon^2 \widehat{\boldsymbol{\omega}}_s = 0, \quad \varepsilon \equiv 1/\sigma \to 0 \\
&\widehat{\boldsymbol{h}}_\tau + \varepsilon[\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{u}}] + \varepsilon^2 \widehat{\boldsymbol{h}}_s = 0 \\
&\operatorname{div} \widehat{\boldsymbol{u}} = 0, \quad \operatorname{div} \widehat{\boldsymbol{h}} = 0
\end{aligned} \tag{3.10}$$

where the subscripts τ and s denote the related partial derivatives.

4. Derivation of the MHD-Drift averaged equation

Let us look for solutions of (3.10) in the form of regular series

$$(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{u}}) = \sum_{k=0}^{\infty} \varepsilon^k (\widehat{\boldsymbol{h}}_k, \widehat{\boldsymbol{u}}_k); \quad \widehat{\boldsymbol{h}}_k, \widehat{\boldsymbol{u}}_k \in \mathbb{H} \cap \mathbb{O}(1), \quad k = 0, 1, 2, \dots \tag{4.1}$$

In this paper we enforce the restriction

$$\overline{\boldsymbol{u}}_0 \equiv 0, \quad \overline{\boldsymbol{h}}_0 \equiv 0 \tag{4.2}$$

which is natural physically if one considers, say, how the secondary vorticity develops on the background of a wave motion. The substitution of (4.1),(4.2) into (3.10) produces the equations of successive approximations. The equations of zero approximation are

$$\widehat{\boldsymbol{\omega}}_{0\tau} = \widetilde{\boldsymbol{\omega}}_{0\tau} = 0; \quad \widehat{\boldsymbol{h}}_{0\tau} = \widetilde{\boldsymbol{h}}_{0\tau} = 0 \tag{4.3}$$

Their unique solution (2.9) is $\widetilde{\boldsymbol{\omega}}_0 \equiv 0$ and $\widetilde{\boldsymbol{h}}_0 \equiv 0$. Taking into account (4.2) we can write

$$\widehat{\boldsymbol{\omega}}_0 \equiv 0, \quad \widehat{\boldsymbol{h}}_0 \equiv 0 \tag{4.4}$$

which means that in zero approximation the flow is potential, purely oscillating, and the magnetic field vanishes. This leads to the similar equations for the first approximation of (3.10),(4.1)-(4.4)

$$\widehat{\boldsymbol{\omega}}_{1\tau} = 0; \quad \widehat{\boldsymbol{h}}_{1\tau} = 0 \tag{4.5}$$

which have the unique solution

$$\widetilde{\boldsymbol{\omega}}_1 \equiv 0, \quad \widetilde{\boldsymbol{h}}_1 \equiv 0, \quad \overline{\boldsymbol{\omega}}_1 = \boxed{?}, \quad \overline{\boldsymbol{h}}_1 = \boxed{?} \tag{4.6}$$

where mean functions remain undetermined. The equations of second approximation that take into account (4.2),(4.4),(4.6) are

$$\widetilde{\boldsymbol{\omega}}_{2\tau} + [\overline{\boldsymbol{\omega}}_1, \widetilde{\boldsymbol{u}}_0] = 0, \quad \widetilde{\boldsymbol{\omega}}_{2\tau} + [\overline{\boldsymbol{\omega}}_1, \widetilde{\boldsymbol{u}}_0] = 0 \tag{4.7}$$

which after $\mathbb{T}$-integration (2.5) yield

$$\widetilde{\boldsymbol{\omega}}_2 = [\widetilde{\boldsymbol{u}}_0^{\tau}, \overline{\boldsymbol{\omega}}_1], \quad \widetilde{\boldsymbol{h}}_2 = [\widetilde{\boldsymbol{u}}_0^{\tau}, \overline{\boldsymbol{h}}_1], \quad \overline{\boldsymbol{\omega}}_2 = \boxed{?}, \quad \overline{\boldsymbol{h}}_2 = \boxed{?} \tag{4.8}$$

The equations of third approximation that take into account (4.2),(4.4),(4.6) are

$$\begin{aligned} &\widetilde{\boldsymbol{\omega}}_{3\tau} + \overline{\boldsymbol{\omega}}_{1s} + [\widehat{\boldsymbol{\omega}}_2, \widetilde{\boldsymbol{u}}_0] + [\overline{\boldsymbol{\omega}}_1, \widehat{\boldsymbol{u}}_1] - [\overline{\boldsymbol{j}}_1, \overline{\boldsymbol{h}}_1] = 0 \\ &\widetilde{\boldsymbol{h}}_{3\tau} + \overline{\boldsymbol{h}}_{1s} + [\widehat{\boldsymbol{h}}_2, \widetilde{\boldsymbol{u}}_0] + [\overline{\boldsymbol{h}}_1, \widehat{\boldsymbol{u}}_1] = 0 \end{aligned} \tag{4.9}$$

The bar-part (2.2) of this system is

$$\begin{aligned} &\overline{\boldsymbol{\omega}}_{1s} + [\overline{\boldsymbol{\omega}}_1, \overline{\boldsymbol{u}}_1] - [\overline{\boldsymbol{j}}_1, \overline{\boldsymbol{h}}_1] + \langle [\widetilde{\boldsymbol{\omega}}_2, \widetilde{\boldsymbol{u}}_0] \rangle = 0 \\ &\overline{\boldsymbol{h}}_{1s} + [\overline{\boldsymbol{h}}_1, \overline{\boldsymbol{u}}_1] + \langle [\widetilde{\boldsymbol{h}}_2, \widetilde{\boldsymbol{u}}_0] \rangle = 0 \end{aligned} \tag{4.10}$$

which can be transformed with the use of (4.8) and (2.14) to the final form

$$\begin{aligned} &\overline{\boldsymbol{\omega}}_{1s} + [\overline{\boldsymbol{\omega}}_1, \overline{\boldsymbol{u}}_1 + \overline{\boldsymbol{V}}_0] - [\overline{\boldsymbol{j}}_1, \overline{\boldsymbol{h}}_1] = 0 \\ &\overline{\boldsymbol{h}}_{1s} + [\overline{\boldsymbol{h}}_1, \overline{\boldsymbol{u}}_1 + \overline{\boldsymbol{V}}_0] = 0 \end{aligned} \tag{4.11}$$

$$\overline{\boldsymbol{V}}_0 \equiv \frac{1}{2} \langle [\widetilde{\boldsymbol{u}}_0, \widetilde{\boldsymbol{u}}_0^{\tau}] \rangle \tag{4.12}$$

If one uses these equations as a closed mathematical model, then all the subscripts and bars can be deleted:

$$\begin{aligned} &\boldsymbol{\omega}_s + [\boldsymbol{\omega}, \boldsymbol{u} + \boldsymbol{V}] - [\boldsymbol{j}, \boldsymbol{h}] = 0 \\ &\boldsymbol{h}_s + [\boldsymbol{h}, \boldsymbol{u} + \boldsymbol{V}] = 0 \end{aligned} \tag{4.13}$$

$$\boldsymbol{V} \equiv \frac{1}{2} \langle [\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{u}}^{\tau}] \rangle \tag{4.14}$$

One can see that:

1. The equation for the oscillating velocity $\widetilde{\boldsymbol{u}}_0$ in our consideration is absent, there are only two restrictions: $\widetilde{\boldsymbol{u}}_0$ is incompressible and potential. Hence the drift velocity $\overline{\boldsymbol{V}}_0$ (4.12) represents a function that is 'external' to the equations.

2. The derived system of equations (4.11) looks similar to the original one (3.1). One may think that (4.11) describes 'just' an additional advection of vorticity and magnetic field by the drift. However, the fact that the averaged vorticity is transported with such an additional velocity is highly non-trivial; in particular, it contains the possibility of the *vortex dynamo* or Langmuir circulations, which we consider below.

5. Stokes drift

Our description of the drift differs from a classical one, therefore we first demonstrate the match of (4.12) with the classical Stokes drift. Let velocity field $\widetilde{\boldsymbol{u}}_0$ and $\widetilde{\boldsymbol{\xi}}_0 \equiv \widetilde{\boldsymbol{u}}_0^\tau$ be

$$\begin{aligned} \widetilde{\boldsymbol{u}}_0(\boldsymbol{x}, s, \tau) &= \overline{\boldsymbol{p}}(\boldsymbol{x}, t)\sin\tau + \overline{\boldsymbol{q}}(\boldsymbol{x}, t)\cos\tau \\ \widetilde{\boldsymbol{\xi}}_0(\boldsymbol{x}, t, \tau) &= -\overline{\boldsymbol{p}}(\boldsymbol{x}, t)\cos\tau + \overline{\boldsymbol{q}}(\boldsymbol{x}, t)\sin\tau \end{aligned} \tag{5.1}$$

with arbitrary $\mathbb{B}$-functions $\overline{\boldsymbol{p}}$ and $\overline{\boldsymbol{q}}$. Straightforward calculations yield

$$[\widetilde{\boldsymbol{u}}_0, \widetilde{\boldsymbol{\xi}}_0] = [\overline{\boldsymbol{p}}, \overline{\boldsymbol{q}}] \tag{5.2}$$

hence the commutator is surprisingly not oscillating. The drift velocity (4.12) is

$$\overline{\boldsymbol{V}}_0 = \frac{1}{2}\langle[\widetilde{\boldsymbol{u}}_0, \widetilde{\boldsymbol{\xi}}_0]\rangle = \frac{1}{2}[\overline{\boldsymbol{p}}, \overline{\boldsymbol{q}}] \tag{5.3}$$

The *dimensional* solution for a plane potential harmonic travelling wave is

$$\widehat{\boldsymbol{u}}_0^* = U\widetilde{\boldsymbol{u}}_0, \quad , \quad \widetilde{\boldsymbol{u}}_0 = \exp(k^* z^*)\begin{pmatrix} \cos(k^* x^* - \tau) \\ \sin(k^* x^* - \tau) \end{pmatrix} \tag{5.4}$$

where (x^*, z^*) are cartesian coordinates and $k^* = 1/L$ is a wavenumber. In Stokes (1847), Lamb (1932), Debnath (1994) one can see that $U = k^* g^* a^*/\sigma^*$ where a^* and g^* are dimensional spatial wave amplitude and gravity; however these physical details are excessive for our analysis. The dimensionless velocity field (5.4) and $\widetilde{\boldsymbol{\xi}}$ are

$$\widetilde{\boldsymbol{u}}_0 = e^z\begin{pmatrix} \cos(x-\tau) \\ \sin(x-\tau) \end{pmatrix}, \quad \widetilde{\boldsymbol{\xi}}_0 = e^z\begin{pmatrix} -\sin(x-\tau) \\ \cos(x-\tau) \end{pmatrix} \tag{5.5}$$

where both fields (5.5) are unbounded as $z \to \infty$, but it is not essential for our purposes. Fields $\overline{\boldsymbol{p}}(x, z)$, $\overline{\boldsymbol{q}}(x, z)$ (5.1) are

$$\overline{\boldsymbol{p}} = Ae^z\begin{pmatrix} \sin x \\ -\cos x \end{pmatrix}, \quad \overline{\boldsymbol{q}} = Ae^z\begin{pmatrix} \cos x \\ \sin x \end{pmatrix} \tag{5.6}$$

The calculations (with the use of (5.2)) yield

$$\overline{\boldsymbol{V}}_0 = e^{2z}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{5.7}$$

The dimensional version of (5.7) is

$$\overline{\boldsymbol{V}}_0^* = \frac{U^2 k^*}{\sigma^*} e^{2k^* z^*}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{5.8}$$

which coincides with the classical expression for the drift velocity given by Stokes (1847), Lamb (1932), Debnath (1994). To obtain (5.8) one should take into account the difference between the time t and $s = t/\sigma$ (3.9).

6. The averaged Euler's equations and vortex dynamo

A special case of (4.11),(4.13) without a magnetic field is

$$\boldsymbol{\omega}_s + [\boldsymbol{\omega}, \boldsymbol{u} + \boldsymbol{V}] = 0. \quad \operatorname{div} \boldsymbol{u} = 0 \tag{6.1}$$

Different s-independent versions of eqn.(4.11) were derived in the studies of Langmuir circulation by Craik & Leibovich(1976) and for the steady streaming problems by Riley (2001), Ilin & Morgulis (2011); the methods employed by these authors are different and more cumbersome than our method. In order to demonstrate the possibility of vortex dynamo we first notice that eqn.(6.1) can be integrated (in space) as

$$\boldsymbol{u}_s + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \boldsymbol{\omega} \times \boldsymbol{V} = -\nabla p, \quad \nabla \cdot \boldsymbol{u} = 0 \tag{6.2}$$

where $\overline{p}$ is a function of integration and the second equation follows from the continuity equation in (3.1). Let the zero approximation (4.4) represent the plane potential travelling gravity wave (5.5) with the drift velocity (5.7). Let cartesian coordinates (x, y, z) be such that $\boldsymbol{V} = (U, 0, 0)$, $U = e^{2z}$, $\boldsymbol{u} = (u, v, w)$ where all components are x-independent (translationally-invariant), and x, z-variables coincide with ones in (5.5). Then the component form of (6.2) is

$$\begin{aligned}
&u_s + vu_y + wu_z = 0\\
&v_s + uv_y + wv_z - Uu_y = -\overline{p}_y\\
&w_s + vw_y + ww_z - Uu_z = -\overline{p}_z\\
&v_y + w_z = 0
\end{aligned}$$

which can be rewritten as (see Vladimirov (1985), Vladimirov (1985a))

$$\begin{aligned}
&v_s + vv_y + wv_z = -P_y - \rho\Phi_y\\
&w_s + vw_y + ww_z = -P_z - \rho\Phi_z\\
&v_y + w_z = 0\\
&\rho_s + u\rho_x + v\rho_y = 0
\end{aligned} \tag{6.3}$$

where $\rho \equiv u$, $\Phi \equiv U = e^{2z}$, and P is a modified pressure. One can see that (6.3) is mathematically equivalent to the system of equations for an incompressible stratified fluid, written in Boussinesq's approximation. The effective 'gravity field' $\boldsymbol{g} = -\nabla\Phi = (0, 0, -2e^{2z})$ is non-homogeneous that makes the analogy with a 'standard' stratified fluid non-complete. Nevertheless one can see that any increasing function $u(z) \equiv \rho(z)$ (taken from the shear flow $(u, v, w) = (u(z), 0, 0)$) produces 'Taylor instability' of an inversely stratified equilibrium. It leads to the growth of longitudinal vortices and can be connected to Langmuir circulations, see Craik & Leibovich(1976), Leibovich (1983), Craik (1985), Thorpe (2004).

7. Discussion

1. The consideration of this paper is based on the assumption that the enforced frequency σ^* (3.3) of oscillations is higher than all intrinsic frequencies. This frequency appears in our theory via the prescribed potential velocity $\widetilde{\boldsymbol{u}}_0$ (4.3).

2. The prescribed oscillatory velocity $\widetilde{\boldsymbol{u}}_0$ can be caused by different factors. For example, it can be produced by oscillations of boundaries or appear in full viscous theory after the matching of external flow with boundary-layer solution. The latter option is often considered, see Riley (2001), Vladimirov (2008), Ilin & Morgulis (2011).

3. To justify the distinguished limit (3.9) mathematically, one should prove that any

different path $\Omega(\sigma)$ produces an asymptotic solution which contains terms secular in s or does not produce any asymptotic solution at all. The following statement can be proven: for $\overline{\boldsymbol{V}}_0 \neq 0$ (4.12) and the function $\Omega(\sigma) = 1/\omega^\alpha$ (with a constant $\alpha > -1$) all solutions with $\alpha < 1$ contain secular terms, while all equations with $\alpha > 1$ produce a controversial (unsolvable) equations of successive approximations. If $\overline{\boldsymbol{V}}_0 \equiv 0$ then the statement is different but we do not describe it here. The proof is omitted.

4. There is a challenging physical fact to explain: the existence of the distinguished limit (3.9) means that there is a hidden slow time-scale $T_{\text{slow}} = \sigma^* T^2$ in the system.

5. The consideration of translationally-invariant MHD-motion in (4.11) is possible in the spirit of the analogy between MHD flows and stratified flows, see Vladimirov, Moffatt and Ilin (1996).

6. The mathematical justification of the equation (4.11) by the estimation of the error in the original equation (3.1) is easily achievable.

7. The higher approximations of the averaged equation (4.11) can be derived. They are especially useful for the study of motions with $\overline{\boldsymbol{V}}_0 \equiv 0$. In particular, one can show that in this case Langmuir circulations cab be still generated by a similar mechanism.

8. The viscosity and diffusivity can be routinely incorporated in (4.11) as the RHS-terms $\nu\nabla^2\overline{\boldsymbol{\varpi}}_1$ and $\kappa\nabla^2\overline{\boldsymbol{h}}_1$. Accordingly, viscous and diffusion terms will appear in the equations (6.2) and (6.3). At the same time after the incorporation of viscosity one more small parameter appears in the list (3.6), and the distinguished limit should be reconsidered.

9. The incorporation of the density stratification and gravity field into presented theory (or as a separate theory) is straightforward.

10. The abolishing of the requirement of a vanishing mean flow in zeroth approximation (4.2) is also straightforward. However in this case the distinguished limit (3.9) is different and the resulted averaged equations are more complicated than (4.11).

11. For the finite and time-dependent flow domain $\mathcal{D}(t)$ the definition of average (2.2) directly works only if $\boldsymbol{x} \in \mathcal{D}$ at any instant. If at some instant $\boldsymbol{x} \notin \mathcal{D}$ then the theory should include a 'projection' of the boundary condition on the 'undisturbed' boundary. Such a consideration requires the smallness of the amplitude a^*/L of spatial oscillations of fluid particles. However one can see that $a^* \sim \boldsymbol{u}^*/\sigma^*$ and hence $a^*/L \sim 1/\sigma \equiv \varepsilon$ (3.6). Therefore the considering of a time-dependent domain does not introduce any new small parameter, and the distinguished limit (3.9) will stay the same. In particular, it means that our small parameter ε is the same as the dimensional slope of free surface in the theory of Langmuir circulations by Craik & Leibovich(1976).

12. The determining of the function $u(z) \equiv \rho(z)$ in (6.3) (for real Langmuir circulations) requires an additional theory which includes viscosity or turbulent tangential stresses at the free surface. It is interesting, that from this viewpoint an 'unstable stratification' can be continuously generated and amplified by tangential stresses applied at free surface. We do not compete here with the theories by Craik & Leibovich(1976), Leibovich (1983), Thorpe (2004), which describe this phenomenon well.

13. One can suggest that since the equations (4.11) for $\boldsymbol{h} \equiv 0$ do describe a mechanism of vortex dynamo, and the mathematical structure of the full averaged equations (4.11),(4.13) with $\boldsymbol{h} \neq 0$ is similar, then these full equations could also describe a possible mechanism of MHD-dynamo, such as the generation of the magnetic field of the Earth.

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